The Restricted Weyl Group of the Cuntz Algebra and Shift Endomorphisms

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Abstract

It is shown that, modulo the automorphisms which fix the canonical diagonal MASA point-wise, the group of those automorphisms of $\mathcal{O}_n$ which globally preserve both the diagonal and the core UHF-subalgebra is isomorphic, via restriction, with the group of those homeomorphisms of the full one-sided n-shift space which eventually commute along with their inverses with the shift transformation. The image of this group in the outer automorphism group of $\mathcal{O}_n$ can be embedded into the quotient of the automorphism group of the full two-sided n-shift by its center, generated by the shift. If n is prime then this embedding is an isomorphism.

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1 Introduction

Investigations of endomorphisms of the simple C^*-algebras $\mathcal{O}_n$, [13], were initiated by Cuntz in his seminal paper [14]. A key tool for those investigations was provided by a fairly simple observation that unital endomorphisms of $\mathcal{O}_n$ are in an explicit bijective correspondence with unitary elements of this C^*-algebra. In particular, Cuntz studied the subgroup of automorphisms of $\mathcal{O}_n$ which globally preserve the canonical diagonal MASA $\mathcal{D}_n$. He showed that the quotient of this group by its normal subgroup consisting of those automorphisms which fix $\mathcal{D}_n$ point-wise (and this is a maximal abelian subgroup of $\mathrm{Aut}(\mathcal{O}_n)$) is discrete. It is natural to think of this quotient as an analogue of the Weyl group. The Weyl group contains a natural interesting subgroup corresponding to those automorphisms which also globally preserve the core UHF-subalgebra $\mathcal{F}_n$ of $\mathcal{O}_n$. Cuntz proposed in [14] a problem of determining the structure of this restricted Weyl group.

Ever since, endomorphisms of $\mathcal{O}_n$ have been an active area of investigations both intrinsically and in connection to many other areas. Not even attempting any exhausting overview of the relevant literature, let us only mention two pieces of research which influenced our current study the most: Conti-Pinzari work on Jones index for endomorphisms of $\mathcal{O}_n$, [10], and Bratteli-Jørgensen work on iterated function systems and representations of Cuntz algebras, initiated in [6] and [7]. More recently, quite significant progress has been achieved in the study of those endomorphisms which preserve either the core UHF-subalgebra, [11], or the diagonal MASA, [17]. A powerful, novel combinatorial approach to the study of endomorphisms which globally preserve both $\mathcal{F}_n$ and $\mathcal{D}_n$ has been developed in [22], [12] and [9].

The main result of this paper is an explicit and intrinsic description of the restricted Weyl group of the Cuntz algebra $\mathcal{O}_n$ and its image in the outer automorphism group $\mathrm{Out}(\mathcal{O}_n)$ (which we call the restricted outer Weyl group of $\mathcal{O}_n$). The way we achieve this is by analyzing the action of the restricted Weyl group on the diagonal MASA, and by showing that certain class of automorphisms of $\mathcal{D}_n$ admits extensions to permutative automorphisms of $\mathcal{O}_n$ (and thus also of $\mathcal{F}_n$). In general, existence of such extensions is not guaranteed, as demonstrated by [8]. It turns out that the restricted outer Weyl group admits a natural embedding into the group of shift automorphisms of the two-sided full shift (with the mod out center), and for prime n this embedding is actually an isomorphism. These facts have profound implications. For example, they immediately imply that the restricted outer Weyl group is residually finite. For $n = 2$, this result also provides a not unexpected answer to a question left open in [12]. Namely, the restricted outer Weyl group is not amenable (for $n \geq 3$ this was already shown earlier in [22] and [12]). It also follows from our results that the restricted Weyl group of $\mathcal{O}_n$ is big enough to contain a copy of the group of shift automorphisms of the one-sided full n-shift. The groups of shift automorphisms (both for the one-sided and the two-sided shift) have been extensively studied in the literature as they reveal an intriguing and highly nontrivial structure, see [18, 19] and the literature cited therein.

One important outcome of our investigations is a powerful and neat link between two areas: the study of automorphisms of the Cuntz algebras and symbolic dynamics. This kind of interaction was pioneered by Cuntz and Krieger in [15], but we believe its limits

have not been reached yet. On one hand, we expect that the nice results available in the literature on symbolic dynamics could shed new light on some aspects of automorphisms of Cuntz algebras, and possibly even more general classes of C^*-algebras. But perhaps it is even more intriguing to speculate the other way round, as one could hope that the algebraic environment that we unveil could provide new tools and insight for attacking some of the problems in the dynamical systems setting.

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2 Notation and preliminaries

If n is an integer greater than 1, then the Cuntz algebra $\mathcal{O}_n$ is a unital, simple, purely infinite C^*-algebra generated by n isometries $S_1, \ldots, S_n$, satisfying $\sum_{i=1}^n S_i S_i^* = I$, [13]. We denote by W_n^k the set of k-tuples $\mu = (\mu_1, \ldots, \mu_k)$ with $\mu_m \in \{1, \ldots, n\}$, and by W_n the union $\cup_{k=0}^\infty W_n^k$, where $W_n^0 = \{0\}$. We call elements of W_n multi-indices. If $\mu \in W_n^k$ then $|\mu| = k$ is the length of μ. If $\mu = (\mu_1, \ldots, \mu_k) \in W_n$, then $S_\mu = S_{\mu_1} \ldots S_{\mu_k}$ ($S_0 = 1$ by convention) is an isometry with range projection $P_\mu = S_\mu S_\mu^*$. Every word in $\{S_i, S_i^* \mid i = 1, \ldots, n\}$ can be uniquely expressed as $S_\mu S_\nu^*$, for $\mu, \nu \in W_n$ [13, Lemma 1.3].

We denote by $\mathcal{F}_n^k$ the C^*-subalgebra of $\mathcal{O}_n$ spanned by all words of the form $S_\mu S_\nu^*$, $\mu, \nu \in W_n^k$, which is isomorphic to the matrix algebra $M_{n^k}(\mathbb{C})$. The norm closure $\mathcal{F}_n$ of $\cup_{k=0}^\infty \mathcal{F}_n^k$, is the UHF-algebra of type n^∞, called the core UHF-subalgebra of $\mathcal{O}_n$, [13]. It is the fixed point algebra for the gauge action of the circle group $\gamma : U(1) \to \mathrm{Aut}(\mathcal{O}_n)$ defined on generators as $\gamma_t(S_i) = tS_i$. For $k \in \mathbb{Z}$, we denote by $\mathcal{O}_n^{(k)} := \{x \in \mathcal{O}_n : \gamma_t(x) = t^k x\}$, the spectral subspace for this action. In particular, $\mathcal{F}_n = \mathcal{O}_n^{(0)}$. The C^*-subalgebra of $\mathcal{F}_n$ generated by projections P_μ, $\mu \in W_n$, is a MASA (maximal abelian subalgebra) both in $\mathcal{F}_n$ and in $\mathcal{O}_n$. We call it the diagonal and denote $\mathcal{D}_n$. The spectrum of $\mathcal{D}_n$ is naturally identified with X_n — the full one-sided n-shift space. We also set $\mathcal{D}_n^k := \mathcal{D}_n \cap \mathcal{F}_n^k$. Throughout this paper we are interested in the inclusions

$$\mathcal{D}_n \subseteq \mathcal{F}_n \subseteq \mathcal{O}_n.$$

The UHF-subalgebra $\mathcal{F}_n$ posseses a unique normalized trace, denoted τ. We will refer to the restriction of τ to $\mathcal{D}_n$ as to the canonical trace on $\mathcal{D}_n$.

We denote by $\mathcal{S}_n$ the group of those unitaries in $\mathcal{O}_n$ which can be written as finite sums of words, i.e., in the form $u = \sum_{j=1}^m S_{\mu_j} S_{\nu_j}^*$ for some $\mu_j, \nu_j \in W_n$. We also denote $\mathcal{P}_n = \mathcal{S}_n \cap \mathcal{U}(\mathcal{F}_n)$. Then $\mathcal{P}_n = \cup_k \mathcal{P}_n^k$, where $\mathcal{P}_n^k$ are permutation unitaries in $\mathcal{U}(\mathcal{F}_n^k)$. That is, for each $u \in \mathcal{P}_n^k$ there is a unique permutation σ of multi-indices W_n^k such that

$$u = \sum_{\mu \in W_n^k} S_{\sigma(\mu)} S_\mu^*. \tag{1}$$

As shown by Cuntz in [14], there exists the following bijective correspondence between unitaries in $\mathcal{O}_n$ and unital $*$-endomorphisms of $\mathcal{O}_n$ (whose collection we denote

by $\mathrm{End}(\mathcal{O}_n)$). A unitary u in $\mathcal{O}_n$ determines an endomorphism λ_u by

$$\lambda_u(S_i) = uS_i, \quad i = 1, \ldots, n.$$

Conversely, if $\rho : \mathcal{O}_n \to \mathcal{O}_n$ is an endomorphism, then $\sum_{i=1}^n \rho(S_i)S_i^* = u$ gives a unitary $u \in \mathcal{O}_n$ such that $\rho = \lambda_u$. If the unitary u arises from a permutation σ via the formula (1), the corresponding endomorphism will be sometimes denoted by λ_σ. Composition of endomorphisms corresponds to a 'convolution' multiplication of unitaries as follows:

$$\lambda_u \circ \lambda_w = \lambda_{\lambda_u(w)u} \tag{2}$$

We denote by φ the canonical shift:

$$\varphi(x) = \sum_i S_i x S_i^*, \quad x \in \mathcal{O}_n.$$

If we take $u = \sum_{i,j} S_i S_j S_i^* S_j^*$ then $\varphi = \lambda_u$. It is well-known that φ leaves invariant both $\mathcal{F}_n$ and $\mathcal{D}_n$, and that φ commutes with the gauge action γ. We denote by ϕ the standard left inverse of φ, defined as $\phi(a) = \frac{1}{n}\sum_{i=1}^n S_i^* a S_i$.

If $u \in \mathcal{U}(\mathcal{O}_n)$ then for each positive integer k we denote

$$u_k = u\varphi(u)\cdots\varphi^{k-1}(u). \tag{3}$$

We agree that u_k^* stands for $(u_k)^*$. If α and β are multi-indices of length k and m, respectively, then $\lambda_u(S_\alpha S_\beta^*) = u_k S_\alpha S_\beta^* u_m^*$. This is established through a repeated application of the identity $S_i a = \varphi(a)S_i$, valid for all $i = 1, \ldots, n$ and $a \in \mathcal{O}_n$. If $u \in \mathcal{F}_n^k$ for some k then, following [10], we call endomorphism λ_u *localized*.

For algebras $A \subseteq B$ we denote by $\mathcal{N}_B(A) = \{u \in \mathcal{U}(B) : uAu^* = A\}$ the normalizer of A in B and by $A' \cap B = \{a \in A : (\forall b \in B)\ ab = ba\}$ the relative commutant of A in B. We also denote by $\mathrm{Aut}(B, A)$ the collection of all those automorphisms α of B that $\alpha(A) = A$, and by $\mathrm{Aut}_A(B)$ those automorphisms of B which fix A point-wise.

3 The restricted Weyl group of the Cuntz algebra

Let α be an automorphism of $\mathcal{D}_n$. We denote by α_* the corresponding homeomorphism of X_n. We say that α has *property (P)* if there exists m such that for all $k \geq m$ we have

$$\alpha\varphi^k(x) = \varphi^{k-m}\alpha\varphi^m(x) \tag{4}$$

for all $x \in \mathcal{D}_n^1$. That is, α eventually commutes with the shift. Equivalently, $\alpha \in \mathrm{Aut}(\mathcal{D}_n)$ satisfies (P) with m if the endomorphism $\alpha\varphi^m$ commutes with the shift φ. We define

$$\mathfrak{G}_n := \{\alpha \in \mathrm{Aut}(\mathcal{D}_n) : \text{ both } \alpha \text{ and } \alpha^{-1} \text{ have property (P)}\}. \tag{5}$$

Lemma 3.1 *$\mathfrak{G}_n$ is a subgroup of* $\mathrm{Aut}(\mathcal{D}_n)$*.*

Proof. Let α and β belong to $\mathfrak{G}_n$. Take m so large that both α and β satisfy (P) with m. Also, let r be so large that $\beta(\varphi^m(\mathcal{D}_n^1))$ is contained in $\mathcal{D}_n^r$. Since $\mathcal{D}_n^r = \mathcal{D}_n^1\varphi(\mathcal{D}_n^1)\cdots\varphi^{r-1}(\mathcal{D}_n^1)$, there exist linear functionals $f_\mu : \mathcal{D}_n^1 \to \mathbb{C}$, $\mu \in W_n^r$, such that

$$\beta(\varphi^m(x)) = \sum_{\mu\in W_n^r} f_\mu(x)P_{\mu_0}\varphi(P_{\mu_1})\cdots\varphi^{r-1}(P_{\mu_{r-1}}). \tag{6}$$

Thus for $k \geq 2m$ and $x \in \mathcal{D}_n^1$ we have

$$\begin{aligned}
\alpha\beta\varphi^k(x) &= \alpha\varphi^{k-m}\beta\varphi^m(x) \\
&= \alpha\varphi^{k-m}\sum_{\mu\in W_n^r} f_\mu(x)P_{\mu_0}\varphi(P_{\mu_1})\cdots\varphi^{r-1}(P_{\mu_{r-1}}) \\
&= \sum_{\mu\in W_n^r} f_\mu(x)(\alpha\varphi^{k-m}(P_{\mu_0}))\cdots(\alpha\varphi^{k-m+r-1}(P_{\mu_{r-1}})) \\
&= \sum_{\mu\in W_n^r} f_\mu(x)(\varphi^{k-2m}\alpha\varphi^{2m}(P_{\mu_0}))\cdots(\varphi^{k-2m}\alpha\varphi^{2m+r-1}(P_{\mu_{r-1}})) \\
&= \varphi^{k-2m}\alpha\sum_{\mu\in W_n^r} f_\mu(x)\varphi^{2m}(P_{\mu_0})\cdots\varphi^{2m+r-1}(P_{\mu_{r-1}}) \\
&= \varphi^{k-2m}\alpha\beta\varphi^{2m}(x),
\end{aligned}$$

since

$$\sum_{\mu\in W_n^r} f_\mu(x)\varphi^{2m}(P_{\mu_0})\cdots\varphi^{2m+r-1}(P_{\mu_{r-1}}) = \beta\varphi^{2m}(x)$$

by virtue of (6) and property (P) for β. Consequently, the product $\alpha\beta$ satisfies (P) with $2m$, and whence $\mathfrak{G}_n$ is a group. □

We denote $\mathfrak{IG}_n = \{\mathrm{Ad}(u)|_{\mathcal{D}_n} : u \in \mathcal{P}_n\}$. This is a normal subgroup of $\mathfrak{G}_n$, since for $u \in \mathcal{P}_n^k$ we have $\mathrm{Ad}(u)\varphi^k = \varphi^k$. In what follows we agree that $\varphi^0 = \mathrm{id}$.

Lemma 3.2 *If $\alpha \in \mathrm{Aut}(\mathcal{D}_n)$ then $\alpha \in \mathfrak{IG}_n$ if and only if there exist m, k such that $\alpha\varphi^m = \varphi^k$.*

Proof. Let $\alpha \in \mathrm{Aut}(\mathcal{D}_n)$ and let m, k be such that $\alpha\varphi^m = \varphi^k$. Suppose first that $m \geq k$. Then $\alpha\varphi^{m-k}(x) = x$ for all $x \in \varphi^k(\mathcal{D}_n)$. Let $r \geq k$ be so large that $\alpha\varphi^{m-k}(\mathcal{D}_n^k) \subseteq \mathcal{D}_n^r$. Since $\mathcal{D}_n^r = \mathcal{D}_n^k\varphi^k(\mathcal{D}_n^{r-k})$ and $\alpha\varphi^{m-k}$ acts identically on $\varphi^k(\mathcal{D}_n^{r-k})$, it follows that $\alpha\varphi^{m-k}(\mathcal{D}_n^r) \subseteq \mathcal{D}_n^r$. Since the map is injective and the space finite dimensional, we have $\alpha\varphi^{m-k}(\mathcal{D}_n^r) = \mathcal{D}_n^r$. As $\alpha\varphi^{m-k}$ acts identically on $\varphi^r(\mathcal{D}_n)$, it follows that there is a permutation unitary $u \in \mathcal{P}_n^r$ such that $\alpha\varphi^{m-k} = \mathrm{Ad}(u)|_{\mathcal{D}_n}$. In particular, $\alpha\varphi^{m-k}$ is an automorphism of $\mathcal{D}_n$. Thus $m - k = 0$ and hence $\alpha = \mathrm{Ad}(u)|_{\mathcal{D}_n}$. If $m \leq k$ then $\alpha^{-1}\varphi^k = \varphi^m$ and we argue in the same way. This proves one implication. The other one is obvious. □

Lemma 3.3 *If $\alpha \in \mathrm{Aut}(\mathcal{D}_n)$ then there exists $\beta \in \mathrm{Aut}(\mathcal{D}_n)$ such that $\alpha\varphi = \beta\varphi\alpha$ and $\beta(x) = \alpha(x)$ for all $x \in \mathcal{D}_n^1$.*

Proof. Given $x \in \mathcal{D}_n$, there exist unique elements $x_1, \ldots, x_n$ in $\mathcal{D}_n$ such that $x = \sum_{j=1}^n P_j \varphi(x_j)$. We define

$$\beta(x) = \alpha\left(\sum_{j=1}^{n} P_j \varphi(\alpha^{-1}(x_j))\right) . \tag{7}$$

It follows that β is an automorphism of $\mathcal{D}_n$ satisfying the required braiding property. $\square$

If $\alpha \in \mathrm{Aut}(\mathcal{D}_n)$ and $\beta \in \mathrm{Aut}(\mathcal{D}_n)$ is defined by formula (7) then we call β the *braiding automorphism* for α.

As shown in [22, Lemma 5], if $u \in \mathcal{P}_n$ and λ_u is an automorphism of $\mathcal{O}_n$ then the restriction of λ_u to $\mathcal{D}_n$ belongs to $\mathfrak{G}_n$. This can be further generalized, as follows.

Theorem 3.4 *Let $\alpha \in \mathrm{Aut}(\mathcal{D}_n)$. Then the following conditions are equivalent:*

(1) α^{-1} has property (P);

(2) there exists a permutation $u \in \mathcal{P}_n$ such that $\alpha = \lambda_u|_{\mathcal{D}_n}$.

In this case, α extends to an endomorphism of $\mathcal{F}_n$ and both α and α^{-1} are τ-preserving.

Proof. (2) $\Rightarrow$ (1): By assumption, α can be written as $\lim_{h\to\infty} \mathrm{Ad}(u_h)$ (pointwise norm limit) and therefore $\alpha^{-1} = \lim_{h\to\infty} \mathrm{Ad}(u_h)^*$. Now, if $u \in \mathcal{P}_n^r$, one has, for all $k-1 \geq r$ and all $1 \leq i \leq n$,

$$\begin{aligned}
\alpha^{-1}(\varphi^{k-1}(S_iS_i^*)) &= \lim_{h\to\infty} \mathrm{Ad}(u_h)^*(\varphi^{k-1}(S_iS_i^*)) \\
&= \lim_{h\to\infty} \varphi^{h-1}(u^*)\ldots\varphi(u^*)u^*\varphi^{k-1}(S_iS_i^*)u\varphi(u)\ldots\varphi^{h-1}(u) \\
&= \lim_{h\to\infty} \varphi^{h-1}(u^*)\ldots\varphi^{k-r}(u^*)\varphi^{k-r}(\varphi^{r-1}(S_iS_i^*))\varphi^{k-r}(u)\ldots\varphi^{h-1}(u) \\
&= \varphi^{k-r}\Big(\lim_{h\to\infty} \varphi^{h-k+r-1}(u^*)\ldots u^*\varphi^{r-1}(S_iS_i^*)u\ldots\varphi^{h-k+r-1}(u)\Big) \\
&= \varphi^{k-r}(\alpha^{-1}(\varphi^{r-1}(S_iS_i^*))) \, .
\end{aligned}$$

(1) $\Rightarrow$ (2): At first we observe that there exists a permutation unitary $u \in \mathcal{P}_n$ such that the braiding automorphism for α is of the form $\beta = \mathrm{Ad}(u)|_{\mathcal{D}_n}$. Indeed, let α^{-1} satisfy (P) with m. Since $\alpha^{-1}\varphi^{m+1} = \varphi\alpha^{-1}\varphi^m$, we have $\varphi^{m+1} = \alpha\varphi\alpha^{-1}\varphi^m = \beta\varphi^{m+1}$. Therefore, there exists a permutation unitary $u \in \mathcal{P}_n$ such that $\beta = \mathrm{Ad}(u)|_{\mathcal{D}_n}$, by Lemma 3.2.

Now we verify that the unitary u as above satisfies $\alpha = \lambda_u|_{\mathcal{D}_n}$. Note that, by induction, $\alpha\varphi^k = (\beta\varphi)^k\alpha$ for all k. Therefore, $\alpha\varphi^k = (\mathrm{Ad}(u)\varphi)^k\alpha = \mathrm{Ad}(u_k)\varphi^k\alpha$ for all k. In particular, $\alpha\varphi^k(P_i) = \mathrm{Ad}(u_k)\varphi^k\alpha(P_i) = \mathrm{Ad}(u_{k+1})\varphi^k(P_i)$ for all k and i. Now we compute

$$\begin{aligned}
\alpha(P_{i_1}\varphi(P_{i_2})\cdots\varphi^{k-1}(P_{i_k})) &= \alpha(P_{i_1})\alpha\varphi(P_{i_2})\cdots\alpha\varphi^{k-1}(P_{i_k}) \\
&= \mathrm{Ad}(u)(P_{i_1})\,\mathrm{Ad}(u_2)\varphi(P_{i_2})\cdots\mathrm{Ad}(u_k)\varphi^{k-1}(P_{i_k}) \\
&= u_kP_{i_1}u_k^*u_k\varphi(P_{i_2})u_k^*\cdots u_k\varphi^{k-1}(P_{i_k})u_k^* \\
&= \mathrm{Ad}(u_k)(P_{i_1}\varphi(P_{i_2})\cdots\varphi^{k-1}(P_{i_k})).
\end{aligned}$$

That is, $\alpha = \lambda_u|_{\mathcal{D}_n}$, as required.

If the above conditions are satisfied, then in particular α extends to an endomorphism $\lambda_u|_{\mathcal{F}_n}$ of $\mathcal{F}_n$, Thus $\tau\lambda_u|_{\mathcal{F}_n} = \tau$ by uniqueness of trace on $\mathcal{F}_n$ (or the fact that $\lambda_u|_{\mathcal{F}_n}$ is a point-wise limit of inner automorphisms). Hence $\tau\alpha = \tau$ as well. This clearly implies also that $\tau\alpha^{-1} = \tau$. □

The preceding theorem provides a dynamical explanation of the puzzling phenomenon observed in [12] through combinatorial arguments, namely the coexistence of permutative automorphisms and proper endomorphisms of $\mathcal{O}_n$ restricting to automorphisms of the diagonal, in the form of a different dynamics they induce on X_n.

The following proposition is yet another slight generalization of [22, Lemma 5]. We omit the proof.

Proposition 3.5 *Let λ_u be a localized automorphism of $\mathcal{O}_n$. Then, there exists some nonnegative integer m such that, for all $k \geq m$ and all $x \in \mathcal{F}_n^1$,*

$$\lambda_u^{-1} \circ \varphi^k(x) = \varphi^{k-m} \circ \lambda_u^{-1} \circ \varphi^m(x) \ . \tag{8}$$

Theorem 3.6 *If $\alpha \in \mathfrak{G}_n$ then there exists a permutation unitary $u \in \mathcal{P}_n$ such that $\alpha = \lambda_u|_{\mathcal{D}_n}$. If w is any unitary in $\mathcal{O}_n$ such that $\lambda_w|_{\mathcal{D}_n} = \alpha$, then $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$.*

Proof. The first statement follows immediately from Theorem 3.4. Let $w \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_w|_{\mathcal{D}_n} = \alpha$. Then, by the same argument, there exists a permutation unitary $v \in \mathcal{P}_n$ such that $\lambda_v|_{\mathcal{D}_n} = \alpha^{-1}$. Thus $\lambda_w\lambda_v$ is an endomorphism of $\mathcal{O}_n$ acting identically on $\mathcal{D}_n$, and consequently $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$ by [8, Proposition 3.2]. □

Corollary 3.7 *Each $\alpha \in \mathfrak{G}_n$ can be extended to an automorphism of $\mathcal{F}_n$.*

By Theorem 3.4, the restriction $r : \mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \to \mathrm{Aut}(\mathcal{D}_n)$ yields a group embedding $\lambda(\mathcal{P}_n)^{-1} \hookrightarrow \mathfrak{G}_n$, see [14, 12]. Since the restriction map $r : \mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \to \mathrm{Aut}(\mathcal{D}_n)$ is injective on $\lambda(\mathcal{P}_n)^{-1}$, [14, 12], Theorem 3.6 yields the following.

Corollary 3.8 *The restriction $r : \lambda(\mathcal{P}_n)^{-1} \to \mathfrak{G}_n$ is a group isomorphism.*

We recall from [14] that $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$ is the normalizer of $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$ in $\mathrm{Aut}(\mathcal{O}_n)$ and it can be also described as the group $\lambda(\mathcal{N}_{\mathcal{O}_n}(\mathcal{D}_n))^{-1}$ of automorphisms of $\mathcal{O}_n$ induced by elements in the (unitary) normalizer $\mathcal{N}_{\mathcal{O}_n}(\mathcal{D}_n)$. Furthermore, using [21], one can show that $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$ has the structure of a semidirect product $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n) \rtimes \lambda(\mathcal{S}_n)^{-1}$ [12]. In particular, the group $\lambda(\mathcal{P}_n)^{-1}$ is isomorphic with the quotient of the group $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$ by its normal subgroup $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$. We call it the *restricted Weyl group* of $\mathcal{O}_n$, cf. [14, 12]. Thus, the preceding corollary provides in a sense an answer to the question raised by Cuntz in [14].

Corollary 3.9 *Let $u \in \mathcal{S}_n$ be such that $\lambda_u(\mathcal{D}_n) = \mathcal{D}_n$ and $\lambda_u|_{\mathcal{D}_n} \in \mathfrak{G}_n$. Then $u \in \mathcal{P}_n$.*

Proof. By Theorem 3.6, there is $w \in \mathcal{P}_n$ such that $\lambda_w(x) = \lambda_u(x)$ for all $x \in \mathcal{D}_n$. However, the restriction map $r : \lambda(S_n)^{-1} \to \mathrm{Aut}(\mathcal{D}_n)$ is injective, [14, 12]. Thus $u = w$ belongs to $\mathcal{P}_n$. □

4 The restricted outer Weyl group of the Cuntz algebra and shift endomorphisms

Denote by $\mathrm{Inn}\,\lambda(\mathcal{P}_n)^{-1}$ the normal subgroup of $\lambda(\mathcal{P}_n)^{-1}$ consisting of all inner permutative automorphisms $\{\mathrm{Ad}(u) : u \in \mathcal{P}_n\}$. We call the quotient $\lambda(\mathcal{P}_n)^{-1}/\,\mathrm{Inn}\,\lambda(\mathcal{P}_n)^{-1}$ the *restricted outer Weyl group* of $\mathcal{O}_n$. From Theorem 3.6, we get the following.

Corollary 4.1 *The restricted outer Weyl group of $\mathcal{O}_n$ is naturally isomorphic to the quotient $\mathfrak{G}_n/\mathfrak{I}\mathfrak{G}_n$.*

In what follows, if $\alpha \in \mathfrak{G}_n$ then we denote its class in $\mathfrak{G}_n/\mathfrak{I}\mathfrak{G}_n$ by $\overline{\alpha}$. We denote by $\mathrm{End}(\mathcal{D}_n, \varphi)$ the semigroup of unital, injective $*$-homomorphisms from $\mathcal{D}_n$ into itself which commute with the shift. We define E_n as the collection of all those $\alpha \in \mathrm{End}(\mathcal{D}_n, \varphi)$ for which there exists an m and a $\beta \in \mathrm{End}(\mathcal{D}_n, \varphi)$ such that

$$\alpha\beta = \varphi^m. \tag{9}$$

In such a case, we have $\alpha\beta\alpha = \varphi^m\alpha = \alpha\varphi^m$, and thus injectivity of α implies that $\beta\alpha = \varphi^m$ as well. In particular, β itself belongs to E_n. E_n is a subsemigroup of $\mathrm{End}(\mathcal{D}_n, \varphi)$ containing all powers of φ, $\mathfrak{G}^0$, as well as all endomorphisms $\alpha\varphi^m$ with $\alpha \in \mathfrak{G}_n$ and suitably large m.

We note that if $\alpha \in E_n$ then $\alpha_* : X_n \to X_n$ is an open mapping. Indeed, if $U \subseteq X_n$ is open then so is $V = \beta_*^{-1}(U)$, and $U = \beta_*(V)$ since β_* is surjective. Then $\alpha_*(U) = \alpha_*\beta_*(V) = \varphi_*^m(V)$ is open, since φ_*^m is an open mapping.

Lemma 4.2 *If $\alpha \in E_n$ then α_* acts bijectively on periodic words. Consequently, $\tau\alpha(d) = \tau(d)$ for all $d \in \mathcal{D}_n$.*

Proof. Say $x \in X_n$ has period r if $\varphi_*^r(x) = x$. For each r, the set $X_n(r)$ of all points with period r is finite. Furthermore, φ_* restricts to a *bijection* on each $X_n(r)$. If $\mu \in W_n$ and $r \geq |\mu|$ then there are exactly $n^{r-|\mu|}$ words $x \in X_n(r)$ such that $P_\mu(x) = 1$.

Now let α be in E_n, and let $\beta \in E_n$ be such that (9) holds. For each r we have $\alpha_*(X_n(r)) \subseteq X_n(r)$, since α_* commutes with φ_*. If $x, y \in X_n(r)$ and $\alpha_*(x) = \alpha_*(y)$ then also $\varphi_*^m(x) = \beta_*\alpha_*(x) = \beta_*\alpha_*(y) = \varphi_*^m(y)$. Thus $x = y$, since φ_* acts bijectively on $X_n(r)$. Therefore, α_* yields a one-to-one mapping from $X_n(r)$ to itself. By finiteness of $X_n(r)$, this map is bijective.

Now let $\mu \in W_n$ and let $\alpha(P_\mu) = \sum_{j=1}^k P_{\mu_j}$. By subdividing, if necessary, we may assume that each μ_j is of the same length r and that $r \geq |\mu|$. Then the number of words $x \in X_n(r)$ such that $P_\mu(x) = 1$ is the same as the number of words $y \in X_n(r)$ such that $P_{\mu_j}(y) = 1$ for some j. Thus $k = n^{r-|\mu|}$, and consequently

$$\tau\alpha(P_\mu) = \frac{k}{n^r} = \frac{1}{n^{|\mu|}} = \tau(P_\mu).$$

Since μ was arbitrary, the proposition follows. $\square$

The following lemma is due to Mike Boyle, [3], although the proof given below is our own.

Lemma 4.3 (M. Boyle) *If $\alpha \in E_n$ then there exists a k such that the mapping α_* is k-to-one. This k divides a power of n. Thus if n is prime then there exists an r such that α_* is n^r-to-one.*

Proof. Let $\alpha, \beta \in E_n$ satisfy (9). Since $\beta_*\alpha_* = \varphi_*^m$ is n^m-to-one, each $y \in X_n$ has at most n^m inverse images under α_*. Let k be the minimal cardinality of $\alpha_*^{-1}(y)$, and let $\omega \in X_n$ be such that α_*^{-1} has exactly k elements. Then the set $\alpha_*^{-1}(\varphi_*^{-1}(\omega)) = \varphi_*^{-1}(\alpha_*^{-1}(\omega)$ has nk elements. By the minimality of k, this can only happen if each element of $\varphi_*^{-1}(\omega)$ has k inverse images under α_*. Let Ω be the smallest subset of X_n containing ω and closed under taking inverse images of φ_*. It follows from the above that for each $y \in \Omega$ the set $\alpha_*^{-1}(y)$ has k elements. Clearly, Ω is dense in X_n. Now let $y \in X_n$ be arbitrary, and let $\alpha_*^{-1}(y)$ have k' elements. Since α_* is an open mapping, there exists an open subset V of X_n containing y whose each element has at least k' inverse images under α_*. Since $V \cap \Omega \neq \emptyset$, we have $k' = k$.

Applying the same reasoning to β instead of α we get an l such that β_* is l-to-one. Thus $kl = n^m$. □

We define an equivalence relation $\sim$ in E_n as follows: $\alpha \sim \beta$ if there exists a k such that either $\alpha = \beta\varphi^k$ or $\alpha\varphi^k = \beta$. Then we set

$$\mathfrak{E}_n = E_n / \sim .$$

By construction, $\mathfrak{E}_n$ is a group. If $\alpha \in E_n$ then we denote its class in $\mathfrak{E}_n$ by $\overline{\alpha}$. Let $\mathrm{Aut}(\Sigma_n)$ denote the group of automorphisms of the full two-sided n-shift, and let $\langle\sigma\rangle$ be its subgroup generated by the two-sided shift σ. The following proposition is well-known, but for completeness we include a proof.

Proposition 4.4 *The groups $\mathfrak{E}_n$ and $\mathrm{Aut}(\Sigma_n)/\langle\sigma\rangle$ are isomorphic.*

Proof. We can realize $\mathcal{D}_n$ as $\otimes_{i\in\mathbb{N}}\mathcal{D}_n^1$. Likewise, we consider $\widetilde{\mathcal{D}}_n := \otimes_{i\in\mathbb{Z}}\mathcal{D}_n^1$, equipped with the two-sided shift automorphism $\widetilde{\varphi}$. The Gelfand spectrum of $\widetilde{\mathcal{D}}_n$ can be identified with Σ_n and the canonical embedding $\mathcal{D}_n \to \widetilde{\mathcal{D}}_n$ corresponds to the canonical surjection $\Sigma_n \to X_n$.

Now, every shift invariant endomorphism ρ of $\mathcal{D}_n$ canonically extends to an endomorphism $\tilde{\rho}$ of $\widetilde{\mathcal{D}}_n$, uniquely determined by the properties of being $\widetilde{\varphi}$-invariant and restricting to ρ on $\mathcal{D}_n \subset \widetilde{\mathcal{D}}_n$. Of course, the shift endomorphism φ extends to $\widetilde{\varphi}$. If ρ is injective then $\tilde{\rho}$ is injective too. Moreover, the map $\rho \mapsto \tilde{\rho}$ gives a semigroup homomorphism from $\mathrm{End}(\mathcal{D}_n, \varphi)$ to $\mathrm{End}(\widetilde{\mathcal{D}}_n, \widetilde{\varphi})$, which is clearly injective.

If $\rho \in E_n$ then it is easy to see that $\tilde{\rho}$ is surjective and thus it is an automorphism of $\widetilde{\mathcal{D}}_n$. Therefore, we get an injective semigroup homomorphism $E_n \to \mathrm{Aut}(\widetilde{\mathcal{D}}_n, \widetilde{\varphi})$. Passing to quotients, the previous map provides a well-defined and injective group homomorphism from $\mathfrak{E}_n$ into $\mathrm{Aut}(\widetilde{\mathcal{D}}_n, \widetilde{\varphi})/\langle\widetilde{\varphi}\rangle \cong \mathrm{Aut}(\Sigma_n)/\langle\sigma\rangle$.

In order to show surjectivity of this map, we observe that given any $\widetilde{\varphi}$-commuting automorphism η of $\widetilde{\mathcal{D}}_n$ there exists a nonnegative integer k such that $\eta' := \widetilde{\varphi}^k\eta|_{\mathcal{D}_n}$ belongs to E_n (use the fact that Σ_n is the Cantor set). Furthermore, thanks to uniqueness of the extension, one has $\widetilde{\varphi}^{-k}\tilde{\eta}' = \eta$ and the proof is complete. □

Theorem 4.5 *There exists an embedding of $\mathfrak{G}_n/\mathfrak{IG}_n$ into $\mathrm{Aut}(\Sigma_n)/\langle\sigma\rangle$. If n is prime then these two groups are isomorphic.*

Proof. By virtue of Proposition 4.4, we may replace $\mathrm{Aut}(\Sigma_n)/\langle\sigma\rangle$ with $\mathfrak{E}_n$.

Let $\alpha \in \mathfrak{G}_n$ satisfy (P) with m. Then $\alpha\varphi^m \in E_n$, and we map α to $\overline{\alpha\varphi^m}$ in $\mathfrak{E}_n$. Clearly, this definition does not depend on the choice of m and thus the map is well defined. One easily checks that this map is a group homomorphism. If $\alpha \in \mathfrak{IG}_n$ then $\alpha\varphi^k = \varphi^k$ for sufficiently large k, and thus the image of such α in $\mathfrak{E}_n$ is the trivial element. Thus, the homomorphism $\mathfrak{G}_n \to \mathfrak{E}_n$ factors though $\mathfrak{G}_n/\mathfrak{IG}_n$, and we get a homomorphism $\mathfrak{G}_n/\mathfrak{IG}_n \to \mathfrak{E}_n$. The latter map is injective. Indeed, let $\alpha \in \mathfrak{G}_n$ satisfy (P) with m and $\overline{\alpha\varphi^m} = \overline{\mathrm{id}}$. Then there is a k such that either $\alpha\varphi^{m+k} = \mathrm{id}$ or $\alpha\varphi^m = \varphi^k$. In either case, $\alpha \in \mathfrak{IG}_n$ by Lemma 3.2.

Now assuming n prime we show that the map $\mathfrak{G}_n/\mathfrak{IG}_n \to \mathfrak{E}_n$ is surjective. Let $\alpha \in E_n$ and let $\beta \in E_n$ be such that identity (9) holds. There is an r such that α_* is n^r-to-one, by Lemma 4.3. Since α_* is an open map onto the Cantor set it admits a continuous section f_1, [20, Corollary 1.4]. Space $X_n \setminus f_1(X_n)$ being open is completely metrizable and thus the restriction of α_* to $X_n \setminus f_1(X_n)$ admits a section f_2. Continuing in this manner, we arrive at a maximal set of independent, continuous sections. For convinience, we label these sections with words from W_n^r. That is, there exist continuous functions $f_\mu : X_n \to X_n$, $\mu \in W_n^r$, such that $\alpha_* f_\mu = \mathrm{id}$ and $\alpha_*^{-1}(x) = \{f_\mu(x) : \mu \in W_n^r\}$ for each $x \in X_n$. Define a mapping $g : X_n \to X_n$ by $g(\mu x) = f_\mu(x)$. Then g is a homeomorphism of X_n. Furthermore, we have $\alpha_* g(\mu x) = \alpha_* f_\mu(x) = x = \varphi_*^r(\mu x)$. Thus $\alpha_* g = \varphi_*^r$ and hence $\alpha_* = \varphi_*^r g^{-1}$. Let $\psi \in \mathrm{Aut}(\mathcal{D}_n)$ be such that $\psi_* = g^{-1}$. Then $\alpha = \psi\varphi^r$ and whence ψ satisfies condition (P). Arguing in the same way, we obtain an $\eta \in \mathrm{Aut}(\mathcal{D}_n)$ such that $\beta = \eta\varphi^t$ for some t. Then $\varphi^m = \alpha\beta = \psi\eta\varphi^{r+t}$. Thus $\psi\eta \in \mathfrak{IG}_n$ by Lemma 3.2. Now ψ^{-1} being a composition of η and an element from $\mathfrak{IG}_n$, itself satisfies condition (P). Consequently, ψ belongs to $\mathfrak{G}_n$. Clearly, the homomorphism $\mathfrak{G}_n/\mathfrak{IG}_n \to \mathfrak{E}_n$ maps $\overline{\psi}$ to $\overline{\alpha}$, and the proof is complete. □

Remarks 4.6 In view of the preceding theorem, the restricted outer Weyl group of $\mathcal{O}_n$ has a number of striking properties known to hold for $\mathrm{Aut}(\Sigma_n)/\langle\sigma\rangle$, see [18, 19] and the references therein. This is immediate at least for n prime. For example, we now know that in the case of $n = 2$ the group $\mathfrak{G}_n/\mathfrak{IG}_n$ is non-amenable (for $n \geq 3$ this has been already observed in [22, 12]). We wonder if the extensive theory of shift automorphisms could not bring new insight into the structural properties of not just Cuntz algebras but also graph algebras and possibly even a larger class of C^*-algebras. In particular, it appears to be an intriguing possibility of translating some features of the beautiful analysis of the action of automorphisms on periodic points, [5], into a more algebraic setting.

In general, when n is not prime, the embedding from Theorem 4.5 is not surjective. This is due to existence of factorizations $\varphi = \alpha\beta$ with $\alpha, \beta \in E_n$ and neither α nor β being an automorphism, [2]. Then it is easy to verify that $\overline{\alpha}$, $\overline{\beta}$ do not belong to the range of the embedding. Nevertheless, even for n not a prime, Theorem 4.5 sheds a lot of light on the structure of the restricted outer Weyl group of $\mathcal{O}_n$. In particular, it implies

that $\mathfrak{G}_n/\mathfrak{I}\mathfrak{G}_n$ is residually finite. As an example, we give an elementary, self-contained proof of the fact that $\mathfrak{E}_n$ is residually finite in Proposition 4.7 below.

Proposition 4.7 *The group $\mathfrak{E}_n$ is residually finite.*

Proof. Let $\mathrm{Per}(X_n)$ denote the set of periodic points in X_n. If $\alpha \in E_n$ then α_* acts bijectively on $\mathrm{Per}(X_n)$, Lemma 4.2. Thus the group $\mathfrak{E}_n$ acts on the orbits of $\mathrm{Per}(X_n)$ under the action of φ_*. The restriction of this action to the orbits contained in $X_n(r)$ yields a homomorphism from $\mathfrak{E}_n$ into a finite permutation group. Thus it suffices to show that if $\alpha \in E_n$ is not a power of the shift then α_* moves at least one orbit.

Let $\alpha \in E_n$. Suppose that α_* fixes every orbit of periodic points under the action of the shift. This means that for each periodic point x there exists k such that $\alpha_*(x) = \varphi_*^k(x)$. We claim that α is a power of the shift. The proof involves the following five steps.

1. There is a map $a : W_n^r \to W_n^1$ such that if $y = \alpha_*(x)$ then $y_1 = a(x_1, \ldots, x_r)$, [16].

2. If $x_1 = \ldots = x_r = j$ then $a(j, \ldots, j) = j$, for otherwise $(j, j, \ldots)$ would be moved by α_*, being a fixed point for φ_*.

3. Let Z be the set of all periodic points of the following form:

$$(x_1, \ldots, x_t, w, \ldots, w)(x_1, \ldots, x_t, w, \ldots, w) \ldots,$$

where $t > 2r$, the number of w's (in one block) is $d > t$, and $w \neq a(x_1, \ldots, x_r)$. Note that Z is dense in X_n. By hypothesis on α_*, the image under α_* of such a periodic point from Z is of the form:

$$\begin{array}{rl} \text{either} & (w, \ldots, w, x_1, \ldots, x_t, w, \ldots, w)(w, \ldots, w, x_1, \ldots, x_t, w, \ldots, w) \ldots \\ \text{or} & (x_p, \ldots, x_t, w, \ldots, x_1, \ldots, x_{p-1})(x_p, \ldots, x_t, w, \ldots, x_1, \ldots, x_{p-1}) \ldots \end{array}$$

However, the former is impossible since $w \neq a(x_1, \ldots, x_r)$ and $t > r$. Furthermore, there is a k in $\{1, \ldots, r\}$ such that $x_k \neq w$, for otherwise $a(x_1, \ldots, x_r) = w$, contrary to the assumption. Consequently, $1 \leq p \leq 2r - 1$.

4. By 3. above, for each $x \in Z$ there is a $0 \leq p \leq 2r - 2$ such that $\alpha_*(x) = \varphi_*^p(x)$. Now let x be an arbitrary point in X_n. Take a sequence $x_k \in Z$ converging to x. Passing to a subsequence we may assume that there is a fixed $p \leq 2r - 2$ such that for each k we have $\alpha_*(x_k) = \varphi_*^p(x_k)$. By continuity, $\alpha_*(x) = \varphi_*^p(x)$.

5. For $j = 0, 1, \ldots, 2r - 2$ let $Y_j = \{x \in X_n : \alpha_*(x) = \varphi_*^j(x)\}$. By 4. above, $X_n = Y_0 \cup \ldots \cup Y_{2r-2}$, and each Y_j is a closed set. Thus there is j such that Y_j contains a non-empty open subset. Then there exists a finite word μ such that for all infinite words x we have $\mu x \in Y_j$ and $\alpha_*(\mu x) = \varphi_*^j(\mu x)$. Since α_* commutes with φ_*, we have

$$\alpha_*(x) = \alpha_* \varphi_*^{|\mu|}(\mu x) = \varphi_*^{|\mu|} \alpha_*(\mu x) = \varphi_*^{j+|\mu|}(\mu x) = \varphi_*^j(x).$$

Since x was arbitrary, $\alpha_* = \varphi_*^j$. □

5 Shift automorphisms

Note that if $\alpha \in \mathrm{Aut}(\mathcal{D}_n)$ satisfies (P) with $m = 0$ then α_* is just an automorphism of the full one-sided n-shift (see [18]). The collection of all such automorphisms constitutes a subgroup of $\mathfrak{G}_n$, which we denote $\mathfrak{G}_n^0$. In the case of $n = 2$ we have $\mathfrak{G}_2^0 \cong \mathbb{Z}_2$ (generated by the restriction of Archbold's flip-flop, [1]) by [18, Theorem 3.1.1]. But for $n \geq 3$ the group $\mathfrak{G}_n^0$ is infinite (see [18, Chapter 3] and [4, 16]).

Example 5.1 Consider an order two automorphism α of $\mathcal{D}_3$ such that α_* changes subwords 13 and 23 (of any one-sided infinite word) into 23 and 13, respectively, as in [18, Example 3.3.10]. Then α belongs to $\mathfrak{G}_3^0$. Define

$$u = P_{11} + P_{12} + P_{21} + P_{22} + P_3 + S_{23}S_{13}^* + S_{13}S_{23}^*,$$

an order two permutation unitary in $\mathcal{P}_3^2$. We have

$$\begin{aligned} uP_1u^* &= P_{11} + P_{12} + P_{23}, \\ uP_2u^* &= P_{21} + P_{22} + P_{13}, \\ uP_3u^* &= P_3. \end{aligned}$$

One checks that the unitary u commutes with the three minimal projections in $\varphi(u\mathcal{D}_3^1u^*)$. This implies (via an easy inductive argument) that the restriction of λ_u to $\mathcal{D}_3$ commutes with the shift φ. It follows that $\alpha = \lambda_u|_{\mathcal{D}_3}$. □

We note, in passing, that among the three rooted trees associated with automorphism λ_u from Example 5.1 as in [12, Section 4.1], two are of height 2 and one is of height 1 (cf. [9, Section 2.1]).

By Theorem 3.6, we already know that each $\alpha \in \mathfrak{G}_n^0$ may be extended to an automorphism of $\mathcal{O}_n$. Below, we provide an alternative proof of this fact, involving a thourough description of the underlying structure in this specific case and thus leading to a more explicit construction of the required permutation unitary. We will need the following two lemmas. A straightforward proof of the former is omitted.

Lemma 5.2 *Let $\alpha \in \mathfrak{G}_n^0$. If u is a unitary in $\mathcal{O}_n$ such that $uxu^* = \alpha(x)$ for all $x \in \mathcal{D}_n^1$ and $u\varphi^j(\alpha(x))u^* = \varphi^j(\alpha(x))$ for all $x \in \mathcal{D}_n^1$ and all $j = 1, \ldots, k-1$ then $u_kxu_k^* = \alpha(x)$ for all $x \in \mathcal{D}_n^k$.*

Lemma 5.3 *Each $\alpha \in \mathfrak{G}_n^0$ commutes with the left-inverse ϕ of φ.*

Proof. Choose r so large that $\alpha(\mathcal{D}_n^1) \subseteq \mathcal{D}_n^r$. We write $\alpha(P_j) = \sum_{\mu \in W(j)} P_\mu$, where $W(j) \subseteq W_n^r$ is a subset of cardinality n^{r-1}, since $\tau\alpha = \tau$ by Lemma 4.2. Clearly, the sets $\{W(j) : j = 1, \ldots, n\}$ form a partition of W_n^r.

Since α commutes with the shift φ, in order to show that α commutes with its left inverse ϕ it suffices to prove that $1 = \sum_{i=1}^n S_i^*\alpha(P_j)S_i$ for all $j = 1, \ldots, n$. This is equivalent to $\{(\mu_2, \ldots, \mu_r) : \mu \in W(j)\} = W_n^{r-1}$. Now the last claim will follow if we can show that $\alpha(P_j)\varphi(P_{i_1} \cdots \varphi^{r-2}(P_{i_{r-1}})) \neq 0$ for all $i_1, \ldots, i_{r-1}$ and all j. But the last expression is different from 0 if and only if $P_j\varphi(\alpha^{-1}(P_{i_1} \cdots \varphi^{r-2}(P_{i_{r-1}})))$ is, and this is clearly the case. □

Theorem 5.4 *Let α be an automorphism of the full one-sided n-shift, i.e. $\alpha \in \mathfrak{G}_n^0$. Then there exists a permutation unitary $u \in \mathcal{P}_n$ such that $\alpha = \lambda_u|_{\mathcal{D}_n}$.*

Proof. Let $\alpha(P_j) = \sum_{\mu \in W(j)} P_\mu$, as in Lemma 5.3. Then $\{(\mu_2, \ldots, \mu_r) \mid \mu \in W(j)\} = W_n^{r-1}$, for every $j = 1, \ldots, n$. Therefore, there exists a unique $u \in \mathcal{P}_n^r$ such that $uP_\mu u^* = P_{\mu(j)}$ for all $\mu \in W_n^r$, where $\mu_1 = j$ and $\mu(j)$ is the unique multi-index in $W(j)$ such that

$$(\mu_2, \ldots, \mu_r) = (\mu(j)_2, \ldots, \mu(j)_r) \ .$$

It is straightforward to check that the conditions of Lemma 5.2 are satisfied (for arbitrarily large k), and hence $\lambda_u|_{\mathcal{D}_n} = \alpha$. □

Remark 5.5 All the permutation unitaries $u \in \mathcal{P}_n^r$ as in the preceeding theorem have the following general structure: there are n functions $f_1, \ldots, f_n$ from W_n^{r-1} into W_n^1 such that for each $w \in W_n^{r-1}$ the map $j \mapsto f_j(w)$ is a permutation of W_n^1, and moreover for all $\mu \in W_n^r$, $uP_\mu u^* = P_{\sigma_f(\mu)}$, where $\sigma_f((\mu_1, \ldots, \mu_r)) = (f_{\mu_1}(\mu_2, \ldots, \mu_r), \mu_2, \ldots, \mu_r)$. Therefore,

$$u = \sum_{\mu \in W_n^r} S_{\sigma_f(\mu)} S_\mu^* \ .$$

Proposition 5.6 *If $\alpha \in \mathrm{Aut}(X_n)$ is a non-trivial automorphism of the full one-sided n-shift and $\tilde{\alpha}$ is an automorphism of $\mathcal{O}_n$ extending α, then $\tilde{\alpha}$ is outer.*

Proof. Suppose $\tilde{\alpha} = \mathrm{Ad}(u)$ for some $u \in \mathcal{U}(\mathcal{O}_n)$. Then $u \in \mathcal{N}_{\mathcal{O}_n}(\mathcal{D}_n)$ and thus $u = vw$, where $v \in \mathcal{S}_n$ and $w \in \mathcal{U}(\mathcal{D}_n)$, [21]. But $\mathrm{Ad}(w)$ acts identically on $\mathcal{D}_n$. Thus $\tilde{\alpha} = \mathrm{Ad}(v)$. Since $\tau\alpha = \tau$, we must have $v \in \mathcal{P}_n^k$ for some k. But then $\varphi^k(x) = \mathrm{Ad}(v)\varphi^k(x) = \varphi^k \mathrm{Ad}(v)(x)$ for all $x \in \mathcal{D}_n$. Thus $v = 1$ and consequently $\tilde{\alpha} = \mathrm{id}$. □

We have shown, above, that every automorphism of $\mathcal{D}_n$ commuting with the shift extends to an automorphism of $\mathcal{O}_n$. In this subsection we observe that in most cases such an extension cannot commute with the shift φ.

Proposition 5.7 *Let λ_v be an endomorphism of $\mathcal{O}_n$. Then λ_v commutes with φ on $\mathcal{O}_n$ if and only if*

$$v\varphi(v)\theta\varphi(v^*) = \varphi(v)\theta \ , \tag{10}$$

where θ is the flip unitary. If moreover $\lambda_v \in \mathrm{Aut}(\mathcal{O}_n)$ then $\lambda_v\varphi = \varphi\lambda_v$ if and only if $v \in \mathcal{U}(\mathcal{F}_n^1)$, i.e. λ_v is a Bogolubov automorphism of $\mathcal{O}_n$.

Proof. The first statement follows easily from the composition rule of endomorphisms and the fact that $\varphi = \lambda_\theta$, where $\theta \in \mathcal{F}_n^2$.

Since $\lambda_v\varphi = \mathrm{Ad}(v)\varphi\lambda_v$ for all unitaries $v \in \mathcal{U}(\mathcal{O}_n)$, the assumption implies that $\mathrm{Ad}(v)\varphi\lambda_v(x) = \varphi\lambda_v(x)$ for all $x \in \mathcal{O}_n$ and thus, λ_v being an automorphism, $v \in \varphi(\mathcal{O}_n)' \cap \mathcal{O}_n = \mathcal{F}_n^1$. □

Proposition 5.8 *Let $u \in \mathcal{U}(\mathcal{O}_n)$ and suppose that $\lambda_u(\mathcal{F}_n) = \mathcal{F}_n$. If λ_u and φ commute on $\mathcal{F}_n$ then λ_u is a Bogolubov automorphism.*

Proof. Observe that indeed $u \in \mathcal{F}_n$ [11]. By a similar argument as in the previous proposition, we get $\mathrm{Ad}(u)\varphi\lambda_u = \varphi\lambda_u$ on $\mathcal{F}_n$ and therefore $u \in \varphi(\mathcal{F}_n)' \cap \mathcal{F}_n = \mathcal{F}_n^1$. □

In the case $n = 2$, if u is a permutation for which $\lambda_u \in \mathrm{Aut}(\mathcal{O}_2)$ commutes with φ on $\mathcal{D}_2$ then u must be the flip automorphism, i.e. Bogolubov permutation. (And then, in turn, λ_u commutes with φ on the whole of $\mathcal{O}_2$.) However Example 5.1 illustrates that this is not true anymore for $\mathcal{D}_n$ with $n > 2$, i.e. there are permutation automorphisms λ_u (which therefore satisfy automatically $\lambda_u(\mathcal{D}_n) = \mathcal{D}_n$) commuting with φ on $\mathcal{D}_n$ which are not Bogolubov (and thus, by the above proposition, they do not commute with φ on $\mathcal{F}_n$, let alone on $\mathcal{O}_n$).

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